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Solving Multiphoton Jaynes–Cummings Model with Field Nonlinearity by Supersymmetric Unitary Transformation

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We introduce a supersymmetric unitary transformation, to diagonalize the multiphoton Jaynes–Cummings model Hamiltonian based on supersymmetric quantum mechanics theory, that includes any forms of intensity-dependent coupling and field nonlinearity. On doing so, we obtain its eigenvalue and eigenstates, and the time evolution of state vector.

KEY WORDS: Multiphoton JCM; supersymmetric unitary transformation.

The interaction of a single two-level atom with the quantized electromagnetic field of a lossless high-Q cavity is a central problem in cavity quantum electrodynamics. The simplest physical situation can be described by the well-known Jaynes–Cummings model (Jaynes and Cummings, 1963). Many interesting nonclassical effects, such as collapses and revivals of atomic inversion, squeezing of radiation field, etc., have been predicted theoretically and observed experimentally in this model (Meystre, 1992; Raitel *et al.*, 1994). Recently, Fan Hongyi proposed that Jaynes–Cummings model can be solved by supersymmetric unitary transformation (Hongyi, 1997). It is no doubt that this new opinion will further enrich the contents of supersymmetric quantum mechanics. In this paper, we will further use the method to study the multiphoton Jaynes–Cummings models that include any forms of intensity-dependent coupling and field nonlinearity.

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We consider the following Hamiltonian (rotating wave approximation)

$$H = \omega N + \frac{1}{2}\omega_0\sigma_z + R(N) + g[a^{+k} f(N)\sigma_- + f(N)a^k\sigma_+], \qquad (1)$$

where ω is the frequency of the single-mode quantized field, a and a^+ are annihilation and creation operators of the field, respectively; ω_0 is the atomic transition frequency; k can be any nonnegative integer; R and f are Hermitian operators and they are any reasonable functions of the photon number operator $N = a^+ a$; $R(N)$ is field nonlinearity item; and $gf(N)$ denotes intensity-dependent atom–field coupling. This model is a fairly general form for single-mode Jaynes–Cummings model. For instance, if $R(x) = 0$, $k = 1$, and $f(x) = 1/(x + 1)$, it reduces to the generalized Jaynes–Cummings model (Fan and Fan, 1994). As $R = a^{+^2} a^2 = N(N - 1)$, $k = 1$, and $f(x) = 1$, it becomes the Kerr-type micromaser model (Deb and Ray, 1993). It reduces to the Buck–Sukumar model as $R(x) = 0$, $k = 1$, and $f(x) = \sqrt{x + 1}$ (Buck and Sukumar, 1980). If letting $R(x) = 0$, $k = 1$, and $f(x) = \sqrt{[x + 1]/(x + 1)}$, $([x] = (1 - q^x)/(1 - q))$, we recover the q-deformed Jaynes–Cummings models (Crnugelj *et al.*, 1994).

To construct the supersymmetric unitary transformation operator, we first define the supersymmetric transformation generators as follows:

$$Q = a^{+k} f(N)\sigma_-, \qquad (2a)$$

$$Q^+ = f(N)a^k\sigma_+, \qquad (2b)$$

$$N' = f^2(N)\frac{(N + k)!}{N!}\sigma_{++} + f^2(N - k)\frac{N!}{(N - k)!}\sigma_{--}, \qquad (2c)$$

where

$$\sigma_{++} = \sigma_+\sigma_- = \frac{1}{2}(1 + \sigma_z), \qquad \sigma_{--} = \sigma_-\sigma_+ = \frac{1}{2}(1 - \sigma_z). \qquad (3)$$

It is easy to see that $(N',\ Q^+,\ Q)$ form supersymmetric generators and have supersymmetric Lie algebra properties, i.e.

$$Q^2 = Q^{+^2} = 0, \quad [Q^+,\ Q] = N'\sigma_z, \quad \{Q,\ \sigma_z\} = \{Q^+,\ \sigma_z\} = 0,$$

$$N' = \{Q,\ Q^+\}, \quad [N', Q] = [N',\ Q^+] = 0, \quad (Q^+ - Q)^2 = -N', \qquad (4)$$

in which { } denotes the anticommutation bracket. With the help of Eq. (2), Eq. (1) can be written as

$$H = M\omega - \frac{1}{2}k\omega + \frac{1}{2}\Delta\sigma_z + R(M - k\sigma_{++}) + g(Q^+ + Q), \qquad (5)$$

where

$$M = N + k\sigma_{++}, \qquad \Delta = \omega_0 - k\omega. \qquad (6)$$

It can be proved that M is constant of motion and commutes with N', Q, and Q^+, i.e.

$$\lfloor M,\ N' \rfloor = [M,\ Q] = \lfloor M,\ Q^+ \rfloor = 0, \tag{7}$$

Using the property $\sigma_{++}^2 = \sigma_{++}$, we have

$$\begin{aligned} R(M - k\sigma_{++}) &= \sum_{l=0}^{\infty} \frac{R^{(l)}(0)}{l!}(M - k\sigma_{++})^l \\ &= \sum_{l=0}^{\infty} \frac{R^{(l)}(0)}{l!}\Bigg[M^l + lM^{l-1}(-k)\sigma_{++} \\ &\quad + \frac{(l-1)}{2!}M^{l-2}(-1)^2\sigma_{++} + \cdots\Bigg] \\ &= \sum_{l=0}^{\infty} \frac{R^{(l)}(0)}{l!}[M^l + (M-k)^l\sigma_{++} - M^l\sigma_{++}] \\ &= R(M) + [R(M-k) - R(M)]\sigma_{++} = R_1(M) + R_2(M)\sigma_z, \end{aligned} \tag{8}$$

where

$$R_1(M) = \frac{1}{2}[R(M) + R(M-k)], \tag{9}$$

$$R_2(M) = \frac{1}{2}[R(M-k) - R(M)]. \tag{10}$$

With the help of Eqs. (8)–(10), Eq. (5) can be rewritten as

$$H = H_0 + \Delta(M)\sigma_z + g(Q^+ + Q), \tag{11}$$

where

$$H_0 = M\omega - \frac{1}{2}k\omega + R_1(M), \tag{12}$$

$$\Delta(M) = R_2(M) + \frac{1}{2}\Delta. \tag{13}$$

We have now put the general model into the form appropriate for using supersymmetric unitary transformation. By the aid of supersymmetric transformation generators defined above, we construct the supersymmetric unitary transformation operator so that the Hamiltonian in Eq. (11) can be diagonalized. The supersymmetric unitary transformation operator is defined as

$$T = \exp\left[-\frac{\theta}{2}N'^{-\frac{1}{2}}(Q^+ - Q)\right], \tag{14}$$

where θ is function of operators M to be determined later, and $N'^{-\frac{1}{2}}$ is defined as

$$N'^{-\frac{1}{2}} = \left[f^2(N)\frac{(N+k)!}{N!}\right]^{-\frac{1}{2}} \sigma_{++} + \left[f^2(N-k)\frac{N!}{(N-k)!}\right]^{-\frac{1}{2}} \sigma_{--}. \tag{15}$$

With the help of the operator formula $af(N) = f(N+1)a$, $a^+ f(N+1) = f(N)a^+$, and Eq. (7), one can easily verify the following commutation relations

$$\left[N'^{-\frac{1}{2}}, Q\right] = \left[N'^{-\frac{1}{2}}, Q^+\right] = \left[N'^{-\frac{1}{2}}, M\right] = 0, \tag{16}$$

$$[\theta, Q] = [\theta, Q^+] = \left[N'^{-\frac{1}{2}}, \theta\right] = 0. \tag{17}$$

Therefore, Eq. (14) can be expanded to the following form:

$$T = \cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right) N'^{-\frac{1}{2}}(Q^+ - Q). \tag{18}$$

From Eqs. (7), (16), and (17), we have

$$T^{-1}H_0T = H_0, \tag{19}$$

$$T^{-1}(Q+Q^+)T = \cos(\theta)(Q+Q^+) + \sin(\theta)\sqrt{N'}\sigma_z, \tag{20}$$

$$T^{-1}\sigma_z T = \cos(\theta)\sigma_z - \sin(\theta)N'^{-\frac{1}{2}}(Q+Q^+). \tag{21}$$

Therefore,

$$\begin{aligned} H' = T^{-1}HT &= H_0 + g\cos(\theta)(Q+Q^+) + g\sin(\theta)\sqrt{N'}\sigma_z \\ &+ \Delta(M)\left[\cos(\theta)\sigma_z - \sin(\theta)N'^{-\frac{1}{2}}(Q+Q^+)\right]. \end{aligned} \tag{22}$$

If we let

$$tg(\theta) = \frac{g\sqrt{N'}}{\Delta(M)}, \tag{23}$$

we can obtain the diagonalized Hamiltonian as follows

$$H' = T^{-1}HT = H_0 + \sqrt{\Delta^2(M) + N'g^2}\sigma_z \tag{24}$$

It should be pointed out that Eq. (23) should be understood in the sense of eigenvalues and eigenvalue equations for the operators M and N'. The corresponding eigenstates of H' read

$$|\Psi_1'\rangle = |n, +\rangle, \qquad |\Psi_2'\rangle = |n+k, -\rangle, \tag{25}$$

and the eigenequations of H' are given by

$$H'|\Psi_1'\rangle = \left[\left(n + \frac{k}{2}\right)\omega + R_1(n+k) + \sqrt{\Delta^2(n+k) + f^2(n)\frac{(n+k)!}{n!}g^2}\right]|\Psi_1'\rangle, \tag{26}$$

$$H'|\Psi_2'\rangle = \left[\left(n+\frac{k}{2}\right)\omega + R_1(n+k) - \sqrt{\Delta^2(n+k) + f^2(n)\frac{(n+k)!}{n!}g^2}\right]|\Psi_2'\rangle, \tag{27}$$

where

$$\Delta(n+k) = \frac{1}{2}\Delta + R_2(n+k) = \frac{1}{2}[\Delta + R(n) - R(n+k)]. \tag{28}$$

Thus the eigenvalues and eigenstates of the Hamiltonian H are, respectively, given by

$$E_{\pm} = E_0(n) \pm E_1(n), \tag{29}$$

$$|\Psi_1\rangle = T|\Psi_1'\rangle = \cos\left(\frac{\theta}{2}\right)|\Psi_1'\rangle + \sin\left(\frac{\theta}{2}\right)|\Psi_2'\rangle, \tag{30}$$

$$|\Psi_2\rangle = T|\Psi_2'\rangle = \cos\left(\frac{\theta}{2}\right)|\Psi_2'\rangle - \sin\left(\frac{\theta}{2}\right)|\Psi_1'\rangle, \tag{31}$$

where

$$E_0(n) = \left(n+\frac{k}{2}\right)\omega + R_1(n+k), \tag{32}$$

$$E_1(n) = \sqrt{\Delta^2(n+k) + f^2(n)\frac{(n+k)!}{n!}g^2}, \tag{33}$$

$$\cos\left(\frac{\theta}{2}\right) = \frac{1}{\sqrt{2}}\sqrt{1+\frac{\Delta(n+k)}{E_1(n)}}, \tag{34}$$

$$\sin\left(\frac{\theta}{2}\right) = \frac{1}{\sqrt{2}}\sqrt{1-\frac{\Delta(n+k)}{E_1(n)}}. \tag{35}$$

It should be pointed out that the states $|n,-\rangle$ and $(n \le k-1)$, which are not included in Eqs. (30) and (31), are also the eigenstates of H.

Now, we use the eigenstates of H to express the time evolution of wave function from arbitrary initial conditions. Denote by $|\Psi(0)\rangle$ an arbitrary initial condition of the system. We can expand the initial state vector $|\Psi(0)\rangle$ in the following form:

$$|\Psi(0)\rangle = \sum_{n=0}^{\infty}[C_n^+(0)|n,+\rangle + C_n^-(0)|n+k,-\rangle] + \sum_{n=0}^{k-1} D_n(0)|n,-\rangle, \tag{36}$$

where $C_n^{\pm}(0)$ and $D_n(0)$ are complex coefficients satisfying the normalization condition. The wave function at time t is then given by

$$\begin{aligned}|\Psi(t)\rangle &= \exp(-iHt)|\Psi(0)\rangle \\ &= T\exp(-iH't)T^{-1}\sum_{n=0}^{\infty}[C_n^{+}(0)|n,+\rangle + C_n^{-}(0)|n+k,-\rangle] \\ &\quad + \sum_{n=0}^{k-1} D_n(t)|n,-\rangle, \end{aligned} \tag{37}$$

where

$$D_n(t) = \exp\left\{-i\left[n\omega + R(n) - \frac{1}{2}\omega_0\right]t\right\}D_n(0). \tag{38}$$

From Eqs. (24) and (25)–(27), we obtain

$$|\Psi(0)\rangle = \sum_{n=0}^{\infty}[C_n^{+}(t)|n,+\rangle + C_n^{-}(t)|n+k,-\rangle] + \sum_{n=0}^{k-1} D_n(t)|n,-\rangle, \tag{39}$$

where

$$C_n^{+}(t) = [A_n(t)C_n^{+}(0) + B_n(t)C_n^{-}(0)]\exp[-iE_0(n)t], \tag{40}$$

$$C_n^{-}(t) = [B_n(t)C_n^{+}(0) + A_n^{*}(t)C_n^{-}(0)]\exp[-iE_0(n)t], \tag{41}$$

$$A_n(t) = \cos[E_1(n)t] - \frac{i\Delta(n+k)}{E_1(n)}\sin[E_1(n)t], \tag{42}$$

$$B_n(t) = -i\frac{gf(n)}{E_1(n)}\sqrt{\frac{(n+k)!}{n!}}\sin[E_1(n)t]. \tag{43}$$

In short, the main result of this paper is the construction of a supersymmetric unitary transformation to diagonalize the Hamiltonian of the multiphoton Jaynes–Cummings model that include any forms of intensity-dependent coupling and field nonlinearity. This method, allowing a unified treatment of different interaction models, seems to be particularly promising because of the possibility to single out worth-noting physical features explicitly related to specific forms of the atom–cavity mode coupling. Moreover, we have obtained the eigenvalue, eigenstates, and time evolution of the state vector of the system. These general results immediately give the solution to any specific forms of intensity-dependent coupling and field nonlinearity, and will facilitate the subsequent investigations of the nonlinear dynamical and statistical properties of the system.

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